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The Exactly Solvable Pöschl-Teller Potential

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We revisit a well known quantum mechanical problem, the trigonometric Pöschl-Teller potential, which is an exactly solvable one-dimensional problem. The potential appears in many physical systems of study and thus the technique of solution is interesting and important for students and readers to know. We wish to elaborate on this in this article.

I. INTRODUCTION

An interesting problem in one-dimensional quantum mechanics is the trigonometric Pösch-Teller potential. This appears in several interesting scenarios in physics [1–3]. Although it is complicated to look at, it turns out that it is exactly solvable. Here we follow Ref. [4] and elaborate the technique of solution for the purpose of the reader. The form of the potential is given by

$$V(x) = \frac{V_0}{2}\left\{\frac{\chi(\chi-1)}{\sin^2\alpha x} + \frac{\lambda(\lambda-1)}{\cos^2\alpha x}\right\} \tag{1}$$

with $\chi > 1$ and $\lambda > 1$ and $V_0 = \hbar^2\alpha^2/m$. This potential is bounded on both sides by singularities at $x = 0$ and $x = \frac{\pi}{2\alpha}$. As $\chi \to 1$, the singularity becomes less and less pronounced at $x = 0$, until at $\chi = 1$, it vanishes. In that case the potential becomes

$$V(x) = \frac{V_0}{2}\frac{\lambda(\lambda-1)}{\cos^2\alpha x} = \frac{V_0}{8}\left[\frac{1}{\sin^2(\frac{\alpha x}{2}-\frac{\pi}{4})} + \frac{1}{\cos^2(\frac{\alpha x}{2}-\frac{\pi}{4})}\right], \tag{2}$$

which again has the same form, with singularities at $\alpha x = \pm\frac{\pi}{2}$, and $\chi = \lambda$. Thus in the $\chi = 1$ case, the two potential 'holes' between $-\frac{\pi}{2}$ and 0 and 0 and $\frac{\pi}{2}$ unite into one large hole. Thus the potential form remains same with redefinitions $\alpha \to \alpha/2$ and $V_0 \to V_0/4$. More over, the potential is clearly periodic but for solving the Schrodinger equation, it plays no role, since the barriers are impenetrable. Thus we pick a single hole, say the interval $0 \le x \le \frac{\pi}{2\alpha}$ and solve the Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x). \tag{3}$$

$\psi(x)$ is obviously a bound state since we have infinite barriers at 0 and $\frac{\pi}{2\alpha}$. Eq. (4) can be rewritten as

$$\frac{d^2\psi(x)}{dx^2} - \frac{2m}{\hbar^2}V(x)\psi(x) + \frac{2m}{\hbar^2}E\psi(x) = 0. \tag{4}$$

II. SOLUTION BY HYPERGEOMETRIC SERIES

Making the following change of variables : $y = \sin^2 \alpha x$, Eq. (4) becomes

$$y(1-y)\frac{d^2\tilde{\psi}(y)}{dy^2} + \left(\frac{1}{2} - y\right)\frac{d\tilde{\psi}(y)}{dy} + \frac{1}{4}\left(\frac{\kappa^2}{\alpha^2} - \frac{\chi(\chi-1)}{y} - \frac{\lambda(\lambda-1)}{y}\right)\tilde{\psi}(y) = 0, \quad (5)$$

where $\kappa^2 = \frac{2mE}{\hbar^2}$. Here, we denote $\psi(x)$ as $\tilde{\psi}(y)$. This differential equation has three regular singular points at $y = 0, 1, \infty$ and is very close to the hyper-geometric differential equation

$$z(1-z)f''(z) + [c - (a+b+1)z]f'(z) - abf(z) = 0. \quad (6)$$

To we recast them in the above form, assume a series solution about $y = 0$, $\tilde{\psi}(y) = \sum_{n=0}^{\infty} a_n y^{\mu+n}$. Equating the lowest power of y i.e. $y^{\mu-1}$ we get the indicial equation

$$\mu(\mu-1) + \frac{1}{2}\mu - \frac{1}{4}\chi(\chi-1) = 0. \quad (7)$$

Since $a_0 \neq 0$, we get the two roots of the indicial equation : $\mu = \frac{\chi}{2}, \frac{1-\chi}{2}$. Since $\chi > 1$ the second indicial root leads to a solution of the form

$$\lim_{y\to 0} y^{\frac{1-\chi}{2}} \sum_{n=0}^{\infty} a_n y^n \to \infty. \quad (8)$$

Thus, we choose the first indicial root. We can do the same about the point $y = 1$. Assuming a series solution about $y = 1$, $\tilde{\psi}(y) = \sum_{n=0}^{\infty} \bar{a}_n (1-y)^{\nu+n}$, we get the indicial equation

$$\nu(\nu-1) + \frac{\nu}{2} - \frac{1}{4}\lambda(\lambda-1) = 0. \quad (9)$$

We get the following roots $\nu = \frac{\lambda}{2}, \frac{1-\lambda}{2}$. Again, we choose the first root as the second root leads to singular behavior at $y = 1$. Thus, we can again write

$$\tilde{\psi}(y) = y^{\frac{\chi}{2}}(1-y)^{\frac{\lambda}{2}} f(y). \quad (10)$$

Plugging this into E. (5), we get the equation for $f(y)$

$$y(1-y)f''(y) + [(\frac{1}{2} + \chi) - (1 - \chi + \lambda)y]f'(y) + \frac{1}{4}[\frac{\kappa^2}{\alpha^2} - (\chi+\lambda)^2]f(y) = 0. \quad (11)$$

This is finally in Hypergeometric form with $a + b = \chi + \lambda$, $c = \chi + \frac{1}{2}$, $ab = \frac{1}{4}[(\chi+\lambda)^2 - (\chi+\lambda)^2]$. Since the equation is symmetric in a and b we can choose

$$a = \frac{1}{2}(\chi + \lambda + \frac{\kappa}{\alpha})$$
$$b = \frac{1}{2}(\chi + \lambda - \frac{\kappa}{\alpha}). \quad (12)$$

The Hypergeometric equation can now be solved using Frobenius method, with the trial solution

$$f(y) = \sum_{n=0}^{\infty} a_n y^{k+n} \tag{13}$$

from which we get the two possible values of $k = 0, 1 - c$ and the recurrence relation

$$a_{n+1} = a_n \frac{(k+n+a)(k+n+b)}{(k+n+1)(k+n+c)} \tag{14}$$

which yields the solution

$$f_k(y) = a_0 \sum_{n=0}^{\infty} \frac{(k+a)_n (k+b)_n}{(k+1)_n (k+c)_n} y^{k+n}. \tag{15}$$

Here, the Pochammer symbol is defined as

$$\begin{aligned} (k+w)_n &= (k+w)(k+w+1)\ldots(k+w+n-1) \\ &= \frac{\Gamma(k+w+n)}{\Gamma(k+w)}. \end{aligned} \tag{16}$$

Hence, the two solutions are

$$\begin{aligned} f_0(y) &= {}_2F_1(a, b, c; y) \\ f_{1-c}(y) &= y^{1-c} {}_2F_1(a+1-c, b+1-c, 2-c; y). \end{aligned} \tag{17}$$

Now, the full solution in our case is

$$f(y) = c_1 {}_2F_1(a, b, c, y) + c_2 y^{1-c} {}_2F_1(a+1-c, b+1-c, 2-c; y) \tag{18}$$

with $a = \frac{1}{2}(\chi + \lambda + \frac{\kappa}{\alpha})$, $b = \frac{1}{2}(\chi + \lambda - \frac{\kappa}{\alpha})$, $c = \chi + \frac{1}{2}$. Now we want to impose the boundary conditions that the wave function vanishes at $x = 0$ and $x = \frac{\pi}{2\alpha}$ as $V \to \infty$ at those points i. e.

$$\tilde{\psi}(0) = 0, \;\; \tilde{\psi}(1) = 0. \tag{19}$$

Note that

$$\lim_{y \to 0} {}_2F_1(a, b, c; y) \to 1. \tag{20}$$

Thus

$$\lim_{y \to 0} \tilde{\psi}(0) = \lim_{y \to 0} y^{\frac{\chi}{2}} (1-y)^{\frac{\lambda}{2}} f(y) = \lim_{y \to 0} (1-y)^{\frac{\lambda}{2}} \left[c_1 y^{\frac{\chi}{2}} + c_2 y^{\frac{1-\chi}{2}} \right] \to \infty. \tag{21}$$

Since $\chi > 1$, the second term is singular, so we must have $c_2 = 0$. For the case $y \to 1$, it is a bit more subtle. First, we observe the domain convergence of the hypergeometric series using De Alembert's ratio test

$$\lim_{n \to \infty} \left| \frac{(a)_{n+1} (b)_{n+1} y^{n+1}}{(1)_{n+1} (c)_{n+1}} \frac{(1)_n c_n}{(a)_n (b)_n y^n} \right| \to |y|. \tag{22}$$

Thus, the series diverges for $y > 1$. For $y = 1$, since $a_{n+1} \sim a_n$ as $n \to \infty$, the series again diverges. Thus, we can get a solution $\tilde{\psi}(y)$ which vanishes at $y = 1$ if the series truncates. Note that this heuristic argument does not take into account the rate of divergence of the hypergeometric series which plays an overall role required for convergence due to the factor of $(1-y)^{\frac{\lambda}{2}}$ appearing in $\tilde{\psi}(y)$, but it is sufficient for the purpose. Now looking at the Pochammer symbol

$$\begin{aligned}(a)_n &= a(a+1)\ldots(a+n-1)\\(b)_n &= b(b+1)\ldots(b+n-1).\end{aligned} \tag{23}$$

If a and b are 0 or negative integers say $-m$, then the series truncates as the $(m+1)^{\text{th}}$ onward coefficient in the hypergeometric series becomes 0. It $a = -m$, then $b = \chi + \lambda + m$ and vice versa. Since the solution is invariant under exchange of a, b both conditions are equivalent. Finally, we can find the energy eigen value as

$$\begin{aligned}a.b = -m(\chi+\lambda+m) &= \frac{1}{4}\left[(\chi+\lambda)^2 - \frac{\kappa^2}{\alpha^2}\right]\\ \Rightarrow E_m &= \frac{V_0}{2}(\chi+\lambda+2m)^2.\end{aligned} \tag{24}$$

The corresponding eigenfunction function finally takes the form

$$\begin{aligned}\tilde{\psi}(y) &= c_1 y^{\frac{\chi}{2}}(1-y)^{\frac{\lambda}{2}}{}_2F_1(a,b,c;y)\\ &= c_1 \sin^{\chi}(\alpha x)\cos^{\lambda}(\alpha x){}_2F_1(-m,\chi+\lambda+m,\chi+\frac{1}{2};\sin^2\alpha x).\end{aligned} \tag{25}$$

III. CONCLUSION

Here, we provide a review of the method for solving the trigonometric Pösch-Teller potential, which is an exactly solvable one-dimensional quantum mechanical problem. The primary goal of this review was to elaborate the solution to the readers, on how a complicated looking problem can still be solved exactly. In fact, this potential plays an important role in the study of supersymmetric quantum mechanics as well, and the reason for the solvability could be guessed from the nature of the potential. The interested readers may refer to Ref. [5].

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